Exercise 5E

1 **a** $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$

$\mathbf{r}.(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = (\mathbf{i} - \mathbf{j} - \mathbf{k}).(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

$= 2 - 1 - 1$

$= 0$

Therefore:

$\mathbf{r}.(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0$

b $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

$\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$

$\mathbf{r}.(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = (\mathbf{i} + 2\mathbf{j} + \mathbf{k}).(5\mathbf{i} - \mathbf{j} - 3\mathbf{k})$

$= 5 - 2 - 3$

$= 0$

Therefore:

$\mathbf{r}.(5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = 0$

c $\mathbf{a} = 2\mathbf{i} - 3\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

$\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$

$\mathbf{r}.(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = (2\mathbf{i} - 3\mathbf{k}).(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

$= 2 - 12$

$= -10$

Therefore:

$\mathbf{r}.(\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = -10$

d $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}$

$\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$

$\mathbf{r}.(4\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = (4\mathbf{i} - 2\mathbf{j} + \mathbf{k}).(4\mathbf{i} + \mathbf{j} - 5\mathbf{k})$

$= 16 - 2 - 5$

$= 9$

Therefore:

$\mathbf{r}.(4\mathbf{i} + \mathbf{j} - 5\mathbf{k}) = 9$

2 **a** $\mathbf{r}.(2\mathbf{i}+\mathbf{j}+\mathbf{k})=0$

$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).(2\mathbf{i}+\mathbf{j}+\mathbf{k})=0$

$2x+y+z=0$

b $\mathbf{r}.(5\mathbf{i}-\mathbf{j}-3\mathbf{k})=0$

$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).(5\mathbf{i}-\mathbf{j}-3\mathbf{k})=0$

Therefore:

$5x-y-3z=0$

c $\mathbf{r}.(\mathbf{i}+3\mathbf{j}+4\mathbf{k})=-10$

$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).(\mathbf{i}+3\mathbf{j}+4\mathbf{k})=-10$

Therefore:

$x+3y+4z=-10$

d $\mathbf{r}.(4\mathbf{i}+\mathbf{j}-5\mathbf{k})=9$

$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).(4\mathbf{i}+\mathbf{j}-5\mathbf{k})=9$

Therefore:

$4x+y-5z=9$

3 **a** The plane passes through the points $A(1, 2, 0)$, $B(3, 1, -1)$ and $C(4, 3, 2)$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$\quad =2\mathbf{i}-\mathbf{j}-\mathbf{k}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}$

$\quad =3\mathbf{i}+\mathbf{j}+2\mathbf{k}$

So an equation of the plane is:

$\mathbf{r}=\mathbf{i}+2\mathbf{j}+\lambda(2\mathbf{i}-\mathbf{j}-\mathbf{k})+\mu(3\mathbf{i}+\mathbf{j}+2\mathbf{k})$

3 **b** The plane passes through the points $A(3, 4, 1)$, $B(-1, -2, 0)$ and $C(2, 1, 4)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= -4\mathbf{i} - 6\mathbf{j} - \mathbf{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= -\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

So an equation of the plane is:

$\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(-4\mathbf{i} - 6\mathbf{j} - \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

c The plane passes through the points $A(2, -1, -1)$, $B(3, 1, 2)$ and $C(4, 0, 1)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

So an equation of the plane is:

$\mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

d The plane passes through the points $A(-1, 1, 3)$, $B(-1, 2, 5)$ and $C(0, 4, 4)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \mathbf{j} + 2\mathbf{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

So an equation of the plane is:

$\mathbf{r} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

4 a $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

The vector $2\mathbf{i} - \mathbf{j} - \mathbf{k}$ is perpendicular to $\mathbf{n}$

The vector $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is also perpendicular to $\mathbf{n}$

$$\text{So } \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= \mathbf{i}(-2 + 1) - \mathbf{j}(4 + 3) + \mathbf{k}(2 + 3)$$

$$= -\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}$$

So the equation of the plane written in Cartesian form is:

$-x - 7y + 5z + d = 0$

Substituting (1, 2, 0) gives:

$-1 - 14 + 0 + d = 0$

$d = 15$

Therefore:

$-x - 7y + 5z + 15 = 0$

or

$x + 7y - 5z = 15$

4 **b** $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(-4\mathbf{i} - 6\mathbf{j} - \mathbf{k}) + \mu(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

The vector $-4\mathbf{i} - 6\mathbf{j} - \mathbf{k}$ is perpendicular to $\mathbf{n}$

The vector $-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ is also perpendicular to $\mathbf{n}$

$$\text{So } \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -6 & -1 \\ -1 & -3 & 3 \end{vmatrix}$$

$$= \mathbf{i}(-18 - 3) - \mathbf{j}(-12 - 1) + \mathbf{k}(12 - 6)$$

$$= -21\mathbf{i} + 13\mathbf{j} + 6\mathbf{k}$$

So the equation of the plane written in Cartesian form is:

$-21x + 13y + 6z + d = 0$

Substituting (3, 4, 1) gives:

$-63 + 52 + 6 + d = 0$

$d = 5$

Therefore:

$-21x + 13y + 6z + 5 = 0$

or

$21x - 13y - 6z = 5$

c $\mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

The vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ is perpendicular to $\mathbf{n}$

The vector $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is also perpendicular to $\mathbf{n}$

$$\text{So } \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{vmatrix}$$

$$= \mathbf{i}(4 - 3) - \mathbf{j}(2 - 6) + \mathbf{k}(1 - 4)$$

$$= \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$$

So the equation of the plane written in Cartesian form is:

$x + 4y - 3z + d = 0$

Substituting (2, −1, −1) gives:

$2 - 4 + 3 + d = 0$

$d = -1$

Therefore:

$x + 4y - 3z = 1$

4 d $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

The vector $\mathbf{j} + 2\mathbf{k}$ is perpendicular to $\mathbf{n}$

The vector $\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ is also perpendicular to $\mathbf{n}$

So $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 3 & 1 \end{vmatrix}$

$= \mathbf{i}(1 - 6) - \mathbf{j}(0 - 2) + \mathbf{k}(0 - 1)$

$= -5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

So the equation of the plane written in Cartesian form is:

$-5x + 2y - z + d = 0$

Substituting $(-1, 1, 3)$ gives:

$5 + 2 - 3 + d = 0$

$d = -4$

Therefore:

$-5x + 2y - z - 4 = 0$

or

$5x - 2y + z = -4$

5 a Find two directions in the plane and take their vector product to give a normal to the plane.

Two directions are $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$ **and** $\begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$

A normal to the plane $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 0 \\ -1 & 1 & -2 \end{vmatrix} = 6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

Dividing by 2, this gives $3\mathbf{i} + \mathbf{j} - \mathbf{k}$, which is also normal to the plane.
Using $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$, with $\mathbf{n} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{a} = 4\mathbf{j} + 2\mathbf{k}$ (note **a** can be the position vector of any point on the plane), this gives the equation of the plane as:
$\mathbf{r}.(3\mathbf{i} + \mathbf{j} - \mathbf{k}) = (4\mathbf{j} + 2\mathbf{k}).(3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 2$
In Cartesian form this may be written as $3x + y - z = 2$

5 **b** Two directions in the plane are $\begin{pmatrix}2\\3\\-3\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\2\\-3\end{pmatrix}$ **and** $\begin{pmatrix}3\\7\\-2\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\6\\-2\end{pmatrix}$

A normal to the plane $\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-3\\2&6&-2\end{vmatrix}=14\mathbf{i}-4\mathbf{j}+2\mathbf{k}$

Dividing by 2, this gives $7\mathbf{i}-2\mathbf{j}+\mathbf{k}$, which is also normal to the plane.
Using $\mathbf{a}=\mathbf{i}+\mathbf{j}$, this gives the equation of the plane as:
$\mathbf{r}.(7\mathbf{i}-2\mathbf{j}+\mathbf{k})=(\mathbf{i}+\mathbf{j}).(7\mathbf{i}-2\mathbf{j}+\mathbf{k})=5$
In Cartesian form this is $7x-2y+z=5$

c Two directions in the plane are $\begin{pmatrix}2\\0\\-1\end{pmatrix}-\begin{pmatrix}3\\0\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\-1\end{pmatrix}$ **and** $\begin{pmatrix}4\\1\\3\end{pmatrix}-\begin{pmatrix}3\\0\\0\end{pmatrix}=\begin{pmatrix}1\\1\\3\end{pmatrix}$

A normal to the plane $\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&0&-1\\1&1&3\end{vmatrix}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$

Using $\mathbf{a}=3\mathbf{i}$, this gives the equation of the plane as:
$r.(\mathbf{i}+2\mathbf{j}-\mathbf{k})=3\mathbf{i}.(\mathbf{i}+2\mathbf{j}-\mathbf{k})=3$
In Cartesian form this is $x+2y-z=3$

d Two directions in the plane are $\begin{pmatrix}3\\1\\-2\end{pmatrix}-\begin{pmatrix}1\\-1\\6\end{pmatrix}=\begin{pmatrix}2\\2\\-8\end{pmatrix}$ **and** $\begin{pmatrix}4\\1\\0\end{pmatrix}-\begin{pmatrix}1\\-1\\6\end{pmatrix}=\begin{pmatrix}3\\2\\-6\end{pmatrix}$

The normal to the plane is $n=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&2&-8\\3&2&-6\end{vmatrix}=4\mathbf{i}-12\mathbf{j}-2\mathbf{k}$

Dividing by 2, this gives $2\mathbf{i}-6\mathbf{j}-\mathbf{k}$, which is also normal to the plane.
Using $\mathbf{a}=\mathbf{i}-\mathbf{j}+6\mathbf{k}$, this gives the equation of the plane as:
$r.(2\mathbf{i}-6\mathbf{j}-\mathbf{k})=(\mathbf{i}-\mathbf{j}+6\mathbf{k}).(2\mathbf{i}-6\mathbf{j}-\mathbf{k})=2$
In Cartesian form this is $2x-6y-z=2$

6 In these problems, the equation of the line includes the position vector of another point on the plane (for example, take $\lambda = 0$) and includes a direction vector in the plane.

a The line has direction $2\mathbf{i} - \mathbf{k}$, and this is a direction in the plane.
The vector $4\mathbf{i}+3\mathbf{j}+\mathbf{k}-(\mathbf{i}+\mathbf{j}-2\mathbf{k}) = 3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ also lies in the plane.

A normal to the plane $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 3 & 2 & 3 \end{vmatrix} = 2\mathbf{i}-9\mathbf{j}+4\mathbf{k}$

So the equation of the plane is
$r.(2\mathbf{i}-9\mathbf{j}+4\mathbf{k}) = (4\mathbf{i}+3\mathbf{j}+\mathbf{k}).(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})$
$\Rightarrow r.(2\mathbf{i}-9\mathbf{j}+4\mathbf{k}) = 8-27+4$
$\Rightarrow r.(2\mathbf{i}-9\mathbf{j}+4\mathbf{k}) = -15$

b The line has direction $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$
Another vector in the plane is $3\mathbf{i}+5\mathbf{j}+\mathbf{k}-(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = 2\mathbf{i}+3\mathbf{j}+3\mathbf{k}$

A **normal to the plane** $n = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ 2 & 3 & 3 \end{vmatrix} = 12\mathbf{i}-12\mathbf{j}+4\mathbf{k}$

So the equation of the plane is
$r.(12\mathbf{i}-12\mathbf{j}+4\mathbf{k}) = (3\mathbf{i}+5\mathbf{j}+\mathbf{k}).(12\mathbf{i}-12\mathbf{j}+4\mathbf{k})$
$\Rightarrow r.(12\mathbf{i}-12\mathbf{j}+4\mathbf{k}) = 36-60+4$
$\Rightarrow r.(12\mathbf{i}-12\mathbf{j}+4\mathbf{k}) = -20$
$\Rightarrow r.(3\mathbf{i}-3\mathbf{j}+\mathbf{k}) = -5$

c The line has direction $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$
Two points in the plane have position vectors $7\mathbf{i}+8\mathbf{j}+6\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}+\mathbf{k}$, the vector joining these points is $5\mathbf{i}+9\mathbf{j}+5\mathbf{k}$, which lies in the plane.

A **normal to the plane** $n = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 5 & 9 & 5 \end{vmatrix} = -8\mathbf{i}+5\mathbf{j}-\mathbf{k}$

So the equation of the plane is
$r.(-8\mathbf{i}+5\mathbf{j}-\mathbf{k}) = (7\mathbf{i}+8\mathbf{j}+6\mathbf{k}).(-8\mathbf{i}+5\mathbf{j}-\mathbf{k})$
$\Rightarrow r.(-8\mathbf{i}+5\mathbf{j}-\mathbf{k}) = -56+40-6$
$\Rightarrow r.(-8\mathbf{i}+5\mathbf{j}-\mathbf{k}) = -22$
$\Rightarrow r.(8\mathbf{i}-5\mathbf{j}+\mathbf{k}) = 22$

7 The plane contains the line $\frac{x-2}{3}=\frac{y+4}{1}=\frac{z-1}{2}$ and passes through (1, 1, 1)

$x-2=3\lambda \Rightarrow x=2+3\lambda$
$y+4=\lambda \Rightarrow y=-4+\lambda$
$z-1=2\lambda \Rightarrow z=1+2\lambda$
$\mathbf{r}=2\mathbf{i}-4\mathbf{j}+\mathbf{k}+\lambda(3\mathbf{i}+\mathbf{j}+2\mathbf{k})$

The vector $3\mathbf{i}+\mathbf{j}+2\mathbf{k}$ is perpendicular to $\mathbf{n}$

The vector $\mathbf{i}+\mathbf{j}+\mathbf{k}-(2\mathbf{i}-4\mathbf{j}+\mathbf{k})=-\mathbf{i}+5\mathbf{j}$ is also perpendicular to $\mathbf{n}$

So $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 2\\ -1 & 5 & 0\end{vmatrix}$

$=\mathbf{i}(0-10)-\mathbf{j}(0+2)+\mathbf{k}(15+1)$
$=-10\mathbf{i}-2\mathbf{j}+16\mathbf{k}$

So the equation of the plane written in Cartesian form is:
$-10x-2y+16z+d=0$
Substituting (2, −4, 1) gives:
$-20+8+16+d=0$
$d=-4$
Therefore:
$-10x-2y+16z-4=0$
or
$10x+2y-16z=-4$